RESEARCH REPORT

Trapped Surfaces and the Positivity of Bondi Mass

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TRAPPED SURFACES AND THE POSITIVITY OF BONDI MASS

ABSTRACT

It is shown that the Bondi Mass of an asymptotically flat space-time which satisfies the dominant energy condition and which contains a number of trapped surfaces is positive.

In a recent letter (Ludvigsen and Vickers 1982) we showed that the Bondi Mass of a large class of physically reasonable space-times is necessarily positive. In particular, we proved the following theorem:

THEOREM 1: Let M be an asymptotically flat space-time which satisfies the dominant energy condition. Let I^+ be future null infinity and let N be a non-singular null hypersurface which intersects I^+ in a global space-like cross-section S_∞ and which is bounded in the past by a finite space-like cross-section S_0. Then, if there exists a non-singular, simply connected, compact space-like hypersurface L with boundary S_0, the Bondi momentum $P_{\underline{a}}(S_\infty)$ with respect to S_∞ is future pointing.

A similar result was also recently proved by Horowitz and Perry (1982). Neither of these results are readily applicable to singular or topologically non-trivial space-times such as those containing black holes. In this letter we overcome this difficulty by proving the following variation of theorem 1 which is directly applicable to such a situation.

THEOREM 2: Let M, N, S_∞ and S_0 be as in theorem 1. Then, if there exists a compact, space-like hypersurface L with outer boundary S_0, and several inner boundaries S_i (i=1,2,...,N) which are trapped surfaces (see fig.1), the Bondi momentum $P_{\underline{a}}(S_\infty)$ with respect to S_∞ is future pointing.

We shall prove this theorem by means of spinor methods similar to those used by Witten (1981) in his proof of the positive energy theorem at space-like infinity. We shall, however, use two-spinors rather than four-spinors.

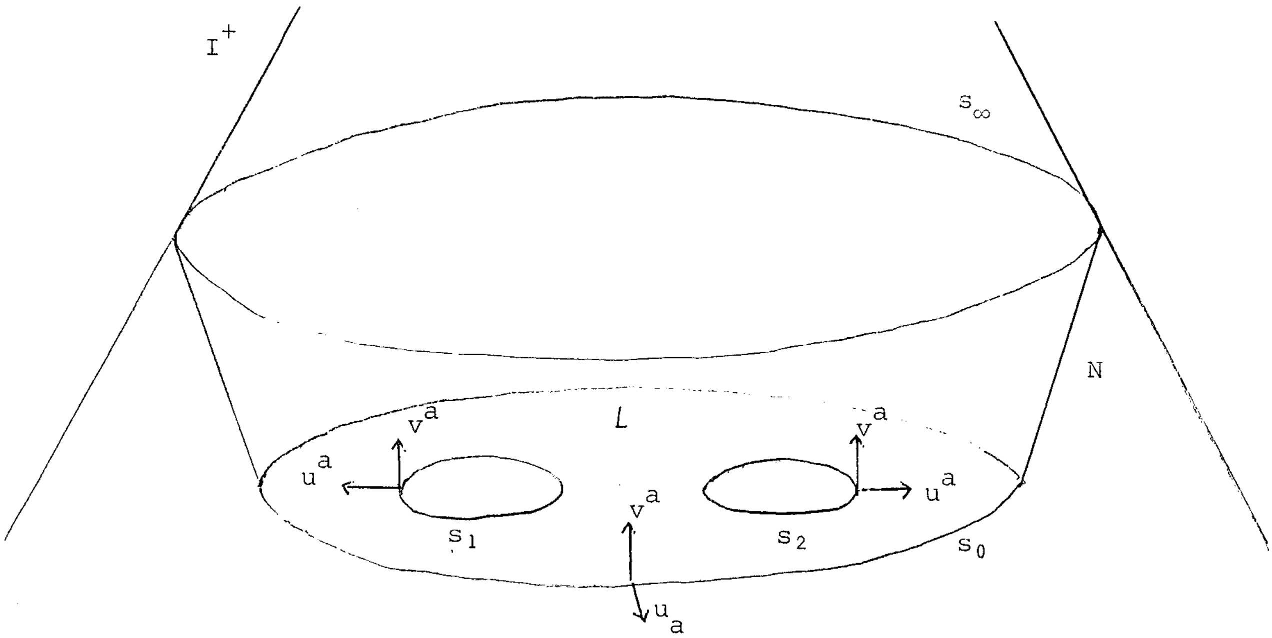

Figure 1: The situation with two trapped surfaces is shown. The hypersurface L has outer boundary S_0 and inner boundaries S_1 and S_2.

The Bondi (1962) four-momentum $P_{\underline{a}}(S_\infty)$ of an asymptotically flat space-time is a four-vector function, defined on the space of all space-like cross sections (cuts) of I^+, which lies in the Minkowski space of BMS translations T. If we let $T = S \times \bar{S}$ where S is the space of two spinors, then, on using the Penrose (1968) abstract index notation, we may write

$$P_{\underline{a}} = P_{\underline{AA}'}$$

where the indices $\underline{A}$ and A' refer to S and $\bar{S}$ respectively. In terms of the standard spin-coefficient notation based on a Bondi coordinate system $(u,r,\zeta,\bar{\zeta})$ and associated spinor dyad field (o_A, ι_A) (see, for example, Exton et al 1969), the Bondi momentum with respect to the origin cut u = 0 may be written as

$$P_{\underline{AA}'} = -\frac{1}{2}\oint (\psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0)\, o_{\underline{A}} o_{\underline{A}'} d\Omega \qquad (1)$$

where the integral is performed over the u = 0 cut of I^+. $o_{\underline{A}}(\zeta,\bar{\zeta})$ is a regular spinor valued function lying in S which has spin weight $\frac{1}{2}$ and which satisfies

$$\eth_0 o_{\underline{A}} = 0$$

where $\eth_0$ is the standard 'edth' operator of Newman and Penrose (1966).

From the above relations we see that $P_{\underline{a}}$ is future pointing if, and only if

$$P_{\underline{AA}'}\lambda^{\underline{A}}\bar{\lambda}^{\underline{A}'} \geqslant 0 \tag{2}$$

for any arbitrary spinor $\lambda^{\underline{A}} \in S$, or, equivalently, that

$$I(S_\infty, \lambda_0^0) := -\frac{1}{2} \oint (\psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0)\lambda_0^0 \bar{\lambda}_0^0 \, d\Omega \geqslant 0 \tag{3}$$

for all regular spin weight $\frac{1}{2}$ functions λ_0^0 satisfying

$$\eth_0 \lambda_0^0 = 0 . \tag{4}$$

In order to prove theorem 2 we start by considering the boundary ∂L of the compact hypersurface L. This consists of several disconnected components; the outer boundary S_0 and N inner boundaries $S_i (i=1,\ldots,N)$. S_0 has topology S^2, and since the inner boundaries are trapped surfaces they too have topology S^2 (Gibbons 1972). Let v^a be the future pointing normal to L, u^a the outgoing normal to S_0 and the ingoing normal to $S_i (i=1,\ldots,N)$, and m^a a complex null vector which lies entirely in ∂L, where the vectors are normalised so that

$$u^a v_a = 0, \ \frac{1}{2} v^a v_a = -\frac{1}{2} u^a u_a = -m^a \bar{m}_a = 1 . \tag{5}$$

Using these vectors we may construct a null tetrad system $(n^a, \ell^a, m^a, \bar{m}^a)$ on ∂L where

$$v^a = \ell^a + n^a, \ u^a = \ell^\alpha - n^a \tag{6}$$

and introduce a spinor dyad (o_A, ι_A) (with $o_A \iota^A = 1$) in the neighbourhood of ∂L which is chosen so that on ∂L we have

$$\left.\begin{array}{l} o_A o_{A'} = \ell_a, o_A \iota_{A'} = m_a, \iota_A o_{A'} = \bar{m}_a, \iota_A \iota_{A'} = n_a \\ \ell_a \nabla^a o_A = n_a \nabla^a o_A = 0 \\ \ell_a \nabla^a \iota_A = n_a \nabla^a \iota_A = 0 \end{array}\right\} \quad (7)$$

Since the S_i are trapped surfaces we have (Penrose 1968).

$$\left.\begin{array}{l} \rho := \iota^A o^{A'} o^B \nabla_{AA'} o_B \geqslant 0 \text{ on } S_i \\ \rho' := -o^A \iota^{A'} \iota^B \nabla_{AA'} \iota_B \geqslant 0 \text{ on } S_i \end{array}\right\} \quad (8)$$

If $(u, r, \zeta, \bar{\zeta})$ is a Bondi type coordinate system in which N is given by u = const., then we may use ζ and $\bar{\zeta}$ to label the points of S_0. Furthermore $\underset{B}{\ell}_a := \nabla_a u$ will be proportional to ℓ_a on S_0. It will be convenient (although not strictly necessary) to choose the hypersurface such that

$$\ell_a = \underset{B}{\ell}_\alpha \, . \quad (9)$$

This simplifies the junction conditions on S_0 and can always be achieved by means of a suitable deformation.

Consider now the following integral over S_0

$$I(S_0, \lambda_A) := \oint_{S_0} F_{ab} d\Sigma^{ab} \quad (10)$$

where $F_{ab} = \phi_{AB} \varepsilon_{A'B'} + \phi_{A'B'} \varepsilon_{AB}$ (11)

$$\phi_{AB} = -\frac{1}{2} \lambda_{C'} \nabla^{C'}{}_{(A} \lambda_{B)} + \frac{1}{2} \lambda_{(A} \nabla^{C'}{}_{B)} \lambda_{C'} \quad (12)$$

and λ_A is some spinor field defined on L.

By writing out $I(S_0, \lambda_A)$ in terms of spin coefficients it may be seen that $I(S_0, \lambda_A)$ depends only upon λ_A and derivatives of λ_A which are intrinsic to S_0; it is thus completely determined by specifying $\lambda_0 = \lambda_A o^A$ and $\lambda_1 = \lambda_A \iota^A$ on S_0. An important property of $I(S_0, \lambda_A)$ which we proved in our earlier letter (Ludvigsen and Vickers 1982) is that

$$I(S_\infty, \lambda_0^0) \geqslant I(S_0, \lambda_A) \quad (13)$$

provided only that $\lambda_0 = \lambda_0^0$.

We now show that for each choice of λ_0^0 satisfying (4) it is possible to choose λ_1 in such a way that

$$I(S_0, \lambda_A) \geqslant 0 .$$

Let λ_A be a solution of the 'Witten equation' on L,

$$D^A_{A'} \lambda_A = 0 \tag{14}$$

where $$D_a = \nabla_a - \frac{1}{2} v_a (v^c \nabla_c) . \tag{15}$$

In terms of the GHP (Geroch et al 1973) spin coefficient based on the spinor dyad (o_A, ι_A) we may write (14) on the boundary ∂L as

$$\tilde{D}\lambda_0 = -2(\eth\lambda_1 + \rho'\lambda_0) \tag{16}$$

$$\tilde{D}\lambda_1 = 2(\bar{\eth}\lambda_0 + \rho\lambda_1) \tag{17}$$

where $\tilde{D} := u^a \nabla_a$.

By using the arguments of Parker and Taubes (1982) (adapted to the compact case) it may be shown that there exists a non-singular solution of (14) which satisfies the boundary condition $\lambda_0 = \lambda_0^0$ on S_0 and

$$\eth\lambda_1 + \rho'\lambda_0 = 0 \text{ on } S_i \quad (i=1,2,\ldots,N) . \tag{18}$$

We now consider the integrals over the inner boundaries

$$I(S_i, \lambda_A) := \oint_{S_i} F_{ab}\, d\Sigma^{ab} . \tag{19}$$

When equation (18) is satisfied these are given by

$$I(S_i, \lambda_A) = \oint_{S_i} (\rho'\lambda_0\bar{\lambda}_0 + \rho\lambda_1\bar{\lambda}_1)\, d\Omega \tag{20}$$

and are therefore positive by the trapped surface condition (8).

Now by Gauss' theorem we have

$$I(S_0, \lambda_A) = \sum_{i=1}^{N} I(S_i, \lambda_A) + \int_L \nabla^b F_{ab}\, dV^a . \tag{21}$$

(The boundary terms having different signs due to the choice of orientation of u^a).

As was shown in an earlier paper the dominant energy condition together with the properties of the Witten equation make the second term on the right positive, while the first term is positive by equation (20). But by equation (13) we have $I(S_\infty, \lambda_0^0) \geqslant I(S_0, \lambda_A)$. We have therefore shown that $I(S_\infty, \lambda_0^0)$ is positive for all regular spin weight $\frac{1}{2}$ functions satisfying $\not{\partial}_0 \lambda_0^0 = 0$, and thus that the Bondi four momentum is future pointing.

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